

# Dimensional versus cut-off renormalization and the nucleon-nucleon potential of the effective field theory

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## Abstract

The role of cut-off and dimensional regularizations is discussed in the context of obtaining a renormalized nucleon-nucleon potential from the chiral Lagrangian formulation of the effective field theory due to Weinberg. Both types of renormalizations are performed for the sum of an attractive delta function and its derivatives. The equivalence between the two forms of regularizations can be established with the use of energy-dependent bare couplings and the explicit forms of these couplings are determined.

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## I. INTRODUCTION

The formulation and study of a nucleon-nucleon potential from a chiral Lagrangian formulation of effective field theory due to Weinberg [1] have become an important topic of investigation in nuclear physics [2]. The nucleon-nucleon potential derived from this effective field theory contains usual finite-range potentials superposed on divergent potentials containing delta function and its derivatives (gradients) and can be written in momentum space as [3–5]

$$\begin{aligned} V(p, q) = & V_f(p, q) + \lambda_1 + \lambda_2(p^2 + q^2) + \lambda_3 p^2 q^2 \\ & + \lambda_4(p^4 + q^4) + \dots, \end{aligned} \tag{1}$$

where  $V_f(p, q)$  represents usual finite-range parts of the potential. The constant term  $\lambda_1$  is the  $\delta$  function. The configuration-space derivatives of the  $\delta$  function appear as powers of momenta in momentum space. These derivatives are the latter terms of (1) with coefficient  $\lambda_i$ ,  $i > 1$ . Apart from the finite-range potential  $V_f(p, q)$ , all other terms of potential (1) are of zero-range and possess ultraviolet divergences in momentum space. There is no convergent calculational scheme with these divergent terms in the Schrödinger framework. Meaningful solution is only obtained after regularization and renormalization of the dynamical equations [3–9].

Lately, there have been a series of studies directed towards a successful regularization and renormalization of potential (1) and similar divergent potentials [3–10]. Three schemes have been used for the purpose: cut-off regularization [3–7,9,10], dimensional regularization [3–5,9], and discretization on the lattice [8]. For the simplest  $\delta$ -function potential all three approaches lead to identical result. Although, the renormalization of the  $\delta$  function term is straightforward and completely under control, inconsistencies appear as the derivatives of the  $\delta$  function are included in the renormalization scheme.

Great deal of effort has been spent on the renormalization of the following part of potential (1) [3–5,9]

$$V(p, q) = \lambda_1 + \lambda_2(p^2 + q^2), \quad (2)$$

where we include the sum of the  $\delta$  function and its first derivatives. Both dimensional and cut-off regularizations have been applied for this purpose. However, no satisfactory and consistent renormalized result for the solution has been obtained with potential (2). In some approach the large nucleon-nucleon scattering length sets a new scale at which a perturbative approach breaks down. This scale corresponds to a very low energy.

The dimensional regularization of potential (2) is immediate and simple. But this result could not simulate the general low-energy behavior of the two-body scattering amplitude. The cut-off regularization of potential (2) involves messy algebra and it is difficult to define a bare coupling which will lead to a satisfactory renormalized result consistent with that obtained by dimensional regularization. An excellent account of this controversy is given in Refs. [3].

Two approaches have emerged to avoid these related problems. Kaplan, Savage, and Wise [11] suggested the so-called power divergent subtraction scheme based on a effective Lagrangian with nucleons and pions including contact interaction. The divergent integrals are then treated via dimensional regularization with an unusual subtraction scheme. Consequently, they are able to obtain a convergent perturbative scheme for large scattering length. Subsequently, Cohen and Hansen [12] achieved the same result via a conventional cut-off regularization procedure in configuration space. An interesting discussion on this has appeared in Ref. [13].

Ultarviolet divergences also appear in perturbative quantum field theory and are usually treated by renormalization techniques. There are several variants of renormalization which employ different types of regularizations, such as, the cut-off, and dimensional regularizations. Usually, both regularization schemes lead to the same renormalized result at low energies. The closely related technique of discretization on the lattice in such field theoretic problems also should lead to equivalent results. In view of this the discrepancy and inconsistency found in the renormalization of potential (2) are quite alarming.

The purpose of the present work is to perform a satisfactory renormalization in momentum space which is devoid of the problem and inconsistency mentioned above. The ultraviolet divergences encountered in the solution of the scattering problem with potential (2) are energy dependent. In the usual text-book renormalization problems these divergences are energy independent and one uses an energy-independent bare coupling for renormalization. Recently, we have suggested [6,9] that for a satisfactory and consistent renormalization of potentials with energy-dependent ultraviolet divergences it is advantageous to use energy-dependent bare coupling. By exploiting the flexibility obtained with the use of energy-dependent bare coupling, it is possible to perform a general and consistent renormalization of potential (2). In this work we follow this procedure explicitly, suggest explicit forms for the bare couplings, and perform the renormalization of the potential (2) by employing energy-dependent bare coupling. The present procedure leads to a general scattering solution for a short-range potential and establishes consistency between cut-off and dimensional regularizations.

In Sec. II we renormalize the  $K$  matrix with potential (2) using dimensional and cut-off regularizations and in Sec. III we present a summary of the present investigation.

## II. REGULARIZATION AND RENORMALIZATION

The partial-wave Lippmann-Schwinger equation for the  $K$  matrix  $K(p, q, k^2)$ , at center of mass energy  $k^2$ , is given, in three dimensions, by

$$K(p, k, k^2) = V(p, k) + \mathcal{P} \int q^2 dq V(p, q) G(q; k^2) K(q, k, k^2), \quad (3)$$

with the free Green function  $G(q; k^2) = (k^2 - q^2)^{-1}$ , in units  $\hbar = 2m = 1$ , where  $m$  is the reduced mass;  $\mathcal{P}$  in Eq. (3) denotes principal value prescription for the integral and the momentum-space integration limits are from 0 to  $\infty$ . The (on-shell) scattering amplitude  $t_L(k)$  is defined by

$$\frac{1}{t(k)} = \frac{1}{K(k^2)} + i\frac{\pi}{2}k, \quad (4)$$

where  $K(k^2) \equiv K(k, k, k^2) = -(2/\pi)(\tan \delta/k)$  with  $\delta$  the phase shift. All scattering observables can be calculated using  $t(k)$ . The condition of unitarity is given by

$$\Im t(k) = -\frac{\pi}{2}k|t(k)|^2,$$

where  $\Im$  denotes the imaginary part. Here we employ a  $K$ -matrix description of scattering. Then the renormalization algebra will involve only real quantities and we do not have to worry about unitarity which can be imposed later via Eq. (4). This is the simplest procedure to follow, as all renormalization schemes preserve unitarity.

Now we consider the renormalization of potential (2), which is the sum of a  $\delta$  function and its second derivatives in configuration space. After a straightforward calculation, the on-shell  $K$  matrix for this potential is given by [9]

$$K(k^2) = \frac{2k^2/\lambda_2 + \lambda_1/\lambda_2^2 + I_2(k^2) - 2k^2I_1(k^2) + k^4I_0(k^2)}{1/\lambda_2^2 - 2I_1(k^2)/\lambda_2 + I_1^2(k^2) - \lambda_1I_0(k^2)/\lambda_2^2 - I_0(k^2)I_2(k^2)}, \quad (5)$$

where  $I_L(k^2)$ 's are the following regularized integrals

$$I_L(k^2) = \mathcal{P} \int q^2 dq q^{2L} G_R(q; k^2), \quad (6)$$

where  $G_R(q; k^2)$  is some regularized Green function. One can employ cut-off and dimensionally regularized Green functions to calculate the regularized integral (6). If one employs the following regularized Green function with a sharp cut off  $\Lambda$

$$G_R(q, \Lambda; k^2) = (k^2 - q^2)^{-1}\Theta(\Lambda - q),$$

with  $\Theta(x) = 0$  for  $x < 0$  and  $= 1$  for  $x > 0$ , the integral in Eq. (6) can be analytically evaluated and one has the following renormalized result

$$\begin{aligned}
I_L(k^2, \Lambda) &= - \left[ \sum_{j=0}^L \frac{k^{2(L-j)} \Lambda^{2j+1}}{2j+1} + \frac{k^{2L+1}}{2} \ln \left| \frac{\Lambda - k}{\Lambda + k} \right| \right], \\
&= - \left[ \Lambda + \frac{k}{2} \ln \left| \frac{\Lambda - k}{\Lambda + k} \right| \right], L = 0, \\
&= - \left[ \frac{\Lambda^3}{3} + k^2 \Lambda + \frac{k^3}{2} \ln \left| \frac{\Lambda - k}{\Lambda + k} \right| \right], L = 1, \\
&= - \left[ \frac{\Lambda^5}{5} + \frac{k^2 \Lambda^3}{3} + k^4 \Lambda + \frac{k^5}{2} \ln \left| \frac{\Lambda - k}{\Lambda + k} \right| \right], L = 2.
\end{aligned}$$

These integrals can also be treated by dimensional regularization. The dimensionally regularized results for these integrals in three dimensions are [9]

$$I_L(k^2) = -\frac{1}{2} \Gamma\left(\frac{2L+3}{2}\right) \Gamma\left(\frac{-2L-1}{2}\right) \Re[(-k^2)^{(2L+1)/2}] \quad (7)$$

where  $\Re$  is the real part. As this result involves the real part of an imaginary quantity it is identically zero in three dimensions.

First we perform cut-off regularization in Eq. (5). This expression has to be calculated by introducing a cut off  $\Lambda$  in the Green's function. However, in the end the limit  $\Lambda \rightarrow \infty$  has to be taken. If the couplings  $\lambda_1$  and  $\lambda_2$  are taken to be constants, the ultraviolet divergence appears in Eq. (5) as this limit is taken. For a proper renormalized result to be obtained the parameters  $\lambda_1$  and  $\lambda_2$  of Eq. (5) are to be interpreted as cut-off ( $\Lambda$ ) and energy-dependent bare couplings. This  $\Lambda$  dependence of  $\lambda_1$  and  $\lambda_2$  cancels the divergent parts of the result in (5) when the  $\Lambda \rightarrow \infty$  limit is taken and one obtains a finite  $K$  matrix. It is easy to write the explicit form of the bare couplings, which are

$$\frac{\lambda_1}{\lambda_2^2} = -I_2(k^2, \Lambda) + \Lambda_0(k^2)k^4, \quad (8)$$

$$\frac{1}{\lambda_2} = I_1(k^2, \Lambda) - \Lambda_0(k^2)k^2, \quad (9)$$

where the function  $\Lambda_0(k^2)$  defines the physical scales of the system and characterizes the interaction. The function  $\Lambda_0(k^2)$  has to be chosen appropriately.

If we introduce Eqs. (8) and (9) into Eq. (5), the  $\Lambda \rightarrow \infty$  limit can be taken immediately and one obtains the exact renormalized result

$$K_R(k^2) = -1/\Lambda_0(k^2). \quad (10)$$

The physical scale(s) in  $\Lambda_0(k^2)$  are to be identified with a physical observable(s). If the problem is to be characterized by a single physical scale, for example the scattering length  $a$ , it is appropriate to take  $\Lambda_0(k^2)$  to be independent of  $k^2$ :

$$\Lambda_0(k^2) = -1/a.$$

If it is to be characterized by two physical scales, such as the scattering length  $a$  and effective range  $r_0$ , it is natural to take

$$\Lambda_0(k^2) = -\frac{1}{a} + \frac{1}{2}r_0k^2 + \dots$$

A third and more scales can be accommodated in a similar fashion. Hence at low energies one can accommodate the full effective range expansion. The general solution for the renormalized  $K$  matrix or its inverse at low energies is a polynomial in  $k^2$ .

Next we perform dimensional regularization. From Eq. (7), we find that the dimensional regularization of integrals  $I_L(k^2)$  for  $d = 3$  are all zero. Here we use the energy-dependent bare couplings

$$\frac{\lambda_1}{\lambda_2^2} = +\Lambda_0(k^2)k^4, \quad (11)$$

$$\frac{1}{\lambda_2} = -\Lambda_0(k^2)k^2, \quad (12)$$

in Eq. (5), where the the functions  $\Lambda_0(k^2)$  again define the physical scales of the system and characterizes the interaction. Then we obtain the finite renormalized on-shell  $K$  matrix given by Eq. (10). This result is identical with the result obtained with cut-off regularization above. With the flexibility introduced through the use of energy-dependent bare couplings, the problem and inconsistency mentioned in Sec. I have been avoided. Essentially, we have obtained Eq. (10) just by employing appropriate bare couplings and by introducing subtractions in the divergent integrals without choosing a regularization scheme. In cut-off regularization the integrals  $I_L(k^2, \Lambda)$  are energy-dependent divergent quantities, in dimensional regularization they are zero. This is the only explicit difference between the bare

couplings Eqs. (8) and (9), and Eqs. (11) and (12), respectively. As we never have to choose a specific regularization scheme, the results are identical.

### III. SUMMARY

We have renormalized the  $K$  matrices with potential (2) by cut-off and dimensional regularizations. The solution of the dynamical problem in these cases involves ultraviolet divergences. For this potential all dimensionally regularized divergent integrals over Green functions are identically zero. However, if the divergent integrals are appropriately subtracted with the use of energy-dependent bare couplings without choosing a specific regularization procedure, we obtain a finite result and both cut-off and dimensional regularization schemes lead to equivalent renormalized results. This suggests that once energy-dependent bare couplings are chosen appropriately, the full nucleon-nucleon potential such as (1) can be successfully renormalized using dimensional and cut-off regularization schemes with both schemes leading to the same renormalized result at low energies. However, it may not be easy to perform an analogous renormalization for the full potential (1) analytically. The difficulties that will arise in this task are expected to be only technical in nature, and not questions of principle as commented in other investigations [3,12]. The present study demonstrate that renormalization with conventional cut-off and dimensional regularizations in momentum space are efficient tools for treating divergent potentials in nonrelativistic quantum mechanics provided that appropriate energy-dependent bare couplings are employed.

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